Math 4210 Tutorial 8

1. Prove that

$$\int_0^T B_t \, dB_t = \frac{1}{2} B_T^2 - \frac{1}{2} T.$$

Solution: $\theta_t^n = B_{t_k}$. $t \in [\frac{kT}{n}, \frac{(k+1)T}{n})$. Devide $[0,T]$ into n subinter

Need to show

$$\lim_{n\to\infty} E\left[\int_0^T (B_t - \theta_t^n)^2 dt\right] = 0.$$

$$E\left[\int_0^T (B_t-\theta_t^n)^2 dt\right] = E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (B_t - B_{t_k})^2 dt\right]$$

$$= \sum_{k=0}^{n-1} E\left[\int_{t_k}^{t_{k+1}} (B_t - B_{t_k})^2 dt\right]$$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t - t_k) dt.$$

$$\le \frac{T}{n} \cdot \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} dt = \frac{T^2}{n} \to 0$$

So we have:

$$\int_0^T B_t dB_t = \lim_{n\to\infty} \int_0^T \theta_t^n dB_t$$

$$= \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} B_{t_k} dB_t\right)$$

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} B_{t_k} dB_t = \sum_{k=0}^{n-1} B_{t_k} \left(\int_{t_k}^{t_{k+1}} dB_t\right)$$

$$\sum_{k=0}\int_{t_k} B_{t_k} dB_t = \sum_{k=0} B_{t_k}\left(\int_{t_k} dB_t\right)$$

$$= \sum_{k=0}^{n-1} B_{t_k}\cdot\left(B_{t_{k+1}} - B_{t_k}\right)$$

$$= \sum_{k=0}^{n-1}\left[\frac{1}{2}\left(B_{t_{k+1}}^2 - B_{t_k}^2\right) - \frac{1}{2}\left(B_{t_{k+1}} - B_{t_k}\right)^2\right]$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}\left(B_{t_{k+1}}^2 - B_{t_k}^2\right) - \frac{1}{2}\sum_{k=0}^{n-1}\left(B_{t_{k+1}} - B_{t_k}\right)^2.$$

$$= \frac{1}{2}B_T^2 - \frac{1}{2}\underbrace{\sum_{k=0}^{n-1}\left(B_{t_{k+1}} - B_{t_k}\right)^2}.$$

$\to$ Converges to T in $L^2(\Omega)$

$$= \frac{1}{2}B_T^2 - \frac{1}{2}T.$$

2. Assume u satisfies

$$\begin{cases} \partial_t u + \dfrac{\sigma^2}{2}S^2\partial_{SS}^2 u + rS\partial_S u - ru = 0, \\ u(T, S) = g(S). \end{cases}$$

Compute $u(0, S_0)$.

Solution:

$$x = \ln S$$

$$v(t, x) = u(t, e^x).$$

$$\partial_t v = \partial_t u; \quad \partial_x v = \partial_S u \cdot S, \quad \partial_{xx}^2 v = \partial_{SS}^2 u \cdot S^2 + \partial_S u \cdot S$$

Equation (*) can be rewritten as

$$\partial_t V + \frac{\sigma^2}{2}\cdot(\partial^2_{xx}V - \partial_x V) + r\cdot\partial_x V - r\cdot V = 0.$$

$W(t,x) = e^{-rt}\cdot V(t,x)$ find equation for W.

$$\partial_t V = r\cdot e^{rt}\cdot W(t,x) + e^{rt}\cdot\partial_t W.$$

$$\partial_x V = e^{rt}\cdot\partial_x W$$

$$\partial^2_{xx} V = e^{rt}\partial^2_{xx} W.$$

Then we can rewritte equation (**) as:

$$r\cdot e^{rt}\cdot W(t,x) + e^{rt}\cdot\partial_t W + \frac{\sigma^2}{2}(e^{rt}\cdot\partial^2_{xx}W - \partial_x W) + r\cdot e^{rt}\cdot\partial_x W - r e^{rt} W = 0$$

$$\Rightarrow \begin{cases} \partial_t W + (r - \frac{\sigma^2}{2})\cdot\partial_x W + \frac{\sigma^2}{2}\partial^2_{xx}W = 0. \\ W(T,x) = e^{-rT}\cdot V(T,x). \end{cases}$$

$$X_T = X_0 + (r - \frac{\sigma^2}{2})T + \sigma B_T.$$

$$W(0,x_0) = E[e^{-rT}\cdot g(e^{x_0 + (r-\frac{\sigma^2}{2})\cdot T + \sigma\cdot B_T})]$$

$$= V(0,x_0)$$

$$= u(0, e^{x_0})$$
$$= u(0, S_0)$$

$$\Rightarrow u(0, S_0) = E\left[e^{-rT} \cdot g\left(e^{x_0 + (r - \frac{\sigma^2}{2})T + \sigma B_T}\right)\right]$$
$$= E\left[e^{-rT} \cdot g\left(S_0 \cdot e^{(r - \frac{\sigma^2}{2})T + \sigma B_T}\right)\right].$$